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ScienceDirect
Journal of Sound and Vibration 325 (2009) 206–223
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Nonlocal elasticity theory for vibration of nanoplates

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Received 26 August 2008; received in revised form 5 March 2009; accepted 7 March 2009
Handling Editor: L.G. Tham
Available online 5 April 2009

Abstract

Classical plate theory (CLPT) and first-order shear deformation theory (FSDT) of plates are reformulated using the nonlocal differential constitutive relations of Eringen. The equations of motion of the nonlocal theories are derived. Navier's approach has been used to solve the governing equations for simply supported boundary conditions. Analytical solutions for vibration of the nanoplates such as graphene sheets are presented. Nonlocal theories are employed to bring out the effect of the nonlocal parameter on natural frequencies of the nanoplates. The developed theory has been extended to the analysis of double layered nanoplates. Effect of (i) nonlocal parameter, (ii) length, (iii) height, (iv) elastic modulus and (v) stiffness of Winkler foundation of the plate on nondimensional vibration frequencies are investigated. The theoretical development as well as numerical solutions presented herein should serve as reference for nonlocal theories of nanoplates and nanoshells.

1. Introduction

Nanostructured elements have attracted attention of scientific community due to their superior properties. Conducting experiments with nanoscale size specimens is found to be difficult and expensive. Therefore, development of appropriate mathematical models for nanostructures is an important issue concerning application of nanostructures. Generally, three approaches have been developed to model nanostructures. These approaches are (a) atomistic [1,2], (b) hybrid atomistic–continuum mechanics [3–6] and (c) continuum mechanics. Both atomistic and hybrid atomistic–continuum mechanics are computationally expensive and are not suitable for analyzing large scale systems. Continuum mechanics approach is less computationally expensive than the former two approaches. It has been found that continuum mechanics results are in good agreement with atomistic and hybrid approaches.

Vibration of nanostructures is of great importance in nanotechnology. Understanding vibration behavior of nanostructures is the key step for many NEMS devices like oscillators, clocks and sensor devices. There are already exploratory studies on the continuum models for vibration of carbon nanotubes (CNTs) or similar micro or nanobeam like elements [7–12]. A review related to the importance and modeling of vibration behavior of various nanostructures can be found in Gibson's et al. [13]. In these works it has been suggested

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.03.007

Nomenclature

a, b length and breadth of the plate
C Winkler foundation constant (representing van der Walls forces)
D bending rigidity of the plate
E Young's modulus of the plate material
E_b Young's modulus of the beam material
G shear modulus of the plate material
h thickness of the plate
h_b thickness of the beam
L length (or breadth) of a square plate
L_b length the beam
M_1^{xx}, M_1^{yy}, M_1^{xy} moment resultants
N_0^{xx}, N_0^{yy}, N_0^{xy} in-plane force resultants
q transverse distributed load
$q^1(x), q^2(x)$ equivalent transverse distributed load in the presence of van der Walls forces on the upper and lower plate, respectively
$S(x)$ fourth-order elasticity tensor
u, v displacement of the point (x, y, 0) of plate along x and y-axis, respectively
V_0^{xx}, V_0^{yy} transverse force resultants
w^c, w^f deflections of the single layered plate at point (x, y) calculated using CLPT and FSDT, respectively
w_1^c, w_1^f deflections of the upper plate in double layered plate system calculated using CLPT and FSDT, respectively
w_2^c, w_2^f deflections of the lower plate in double layered plate system calculated using CLPT and FSDT, respectively
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}$ strain tensors
κ^2 shear correction factor
μ nonlocal parameter
v Poisson's ratio of the plate material
ρ density of the plate material
ρ_b density of the beam material
σ^l macroscopic local stress tensor
σ_{xx}^{nl}, σ_{yy}^{nl}, σ_{zz}^{nl}, σ_{xy}^{nl}, σ_{yz}^{nl}, σ_{xz}^{nl} nonlocal stress tensors
ψ_x, ψ_y rotations of a transverse normal in the single layered plate with respect to x and y-axis, respectively
$\psi 1_x$, $\psi 1_y$ rotations of a transverse normal in the upper plate of the double layered plate system with respect to x and y-axis, respectively
$\psi 2_x$, $\psi 2_y$ rotations of a transverse normal in the lower plate of the double layered plate system with respect to x and y-axis, respectively
ω_b natural frequency of the beam
$\bar{\omega}_b$ nondimensional natural frequency of the beam
ω_{mn}^c, ω_{mn}^f natural vibration frequencies of single layered plate calculated using CLPT and FSDT, respectively
$\omega_{mn}^c 1$, $\omega_{mn}^c 2$ natural vibration frequencies of double layered plate calculated using CLPT
$\omega_{mn}^f 1$, $\omega_{mn}^f 2$ natural vibration frequencies of double layered plate calculated using FSDT
∇^2 Laplacian operator in 2D cartesian coordinate system

that nonlocal elasticity theory developed by Eringen [14,15] should be used in the continuum models for accurate prediction of vibration behaviors. This is due to the scale effect of the nanostructures. Importance of accurate prediction of nanostructures' vibration characteristics have been discussed by Gibson et al. [13]. A relevant reference concerning nonlocal theories for bending, buckling and vibration analysis of beams is reported by Reddy [16].

Similar to CNTs, nanoplates possess superior mechanical properties [17,18]. But in contrast to one-dimensional structures, limited work have been found on vibration analysis of two-dimensional nanoplates [18–21]. In the continuum models used in Refs. [18–21] only classical plate theory (CLPT) has been considered for modeling the nanoplates. These mathematical models do not take scale effect into account. It is importance to incorporate nonlocal elasticity theories in the vibration analysis of nanoplates. In the present paper attempt is made to study the vibration of the nanoplates using nonlocal elasticity theory. Both the CLPT and first-order shear deformation theory (FSDT) have been incorporated in the analysis. The developed theory has been extended to the analysis of multilayered nanoplates. Navier's approach has been used to solve the governing equations for simply supported boundary conditions. Effect of (i) nonlocal parameter, (ii) length, (iii) height, (iv) elastic modulus and (v) stiffness of Winkler foundation of the plates on nondimensional vibration frequencies are investigated.

2. Formulation

The coordinate system used for the nanoplate is shown in Fig. 1a. Origin is chosen at one corner of the mid-plane of the plate. The x, y coordinates of the axes are taken along the length and width of the plate. z coordinate is taken along the thickness of the plate. Following stress resultants are used in the present formulation

$$
\begin{aligned}
&N_0^{xx} = \int_{-h/2}^{h/2} \sigma_{xx}^{nl}\,\mathrm{d}z, \quad N_0^{yy} = \int_{-h/2}^{h/2} \sigma_{yy}^{nl}\,\mathrm{d}z, \quad N_0^{xy} = \int_{-h/2}^{h/2} \sigma_{xy}^{nl}\,\mathrm{d}z \\
&V_0^{xx} = \int_{-h/2}^{h/2} \sigma_{xz}^{nl}\,\mathrm{d}z, \quad V_0^{yy} = \int_{-h/2}^{h/2} \sigma_{yz}^{nl}\,\mathrm{d}z, \quad M_1^{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}^{nl}\,\mathrm{d}z \\
&M_1^{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}^{nl}\,\mathrm{d}z, \quad M_1^{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}^{nl}\,\mathrm{d}z
\end{aligned} \tag{1}
$$

Here h denotes the height of the plate. σ_{xx}^{nl}, σ_{yy}^{nl}, σ_{zz}^{nl}, σ_{xy}^{nl}, σ_{yz}^{nl} and σ_{xz}^{nl} represent the nonlocal stress tensors. In classical local elasticity theories, stress at a point depends only on the strain at that point. While in nonlocal elasticity theories it is assumed that the stress at a point depends on the strains at all the points of the continuum. In other words, according to this nonlocal theory strain at a point depends on both stress and spatial derivatives of the stress at that point. According to Eringen [14] the nonlocal constitutive behavior of a Hookean solid is represented by the following differential constitutive relation

$$(1 - \mu\nabla^2)\sigma^{nl} = \sigma^l \tag{2}$$

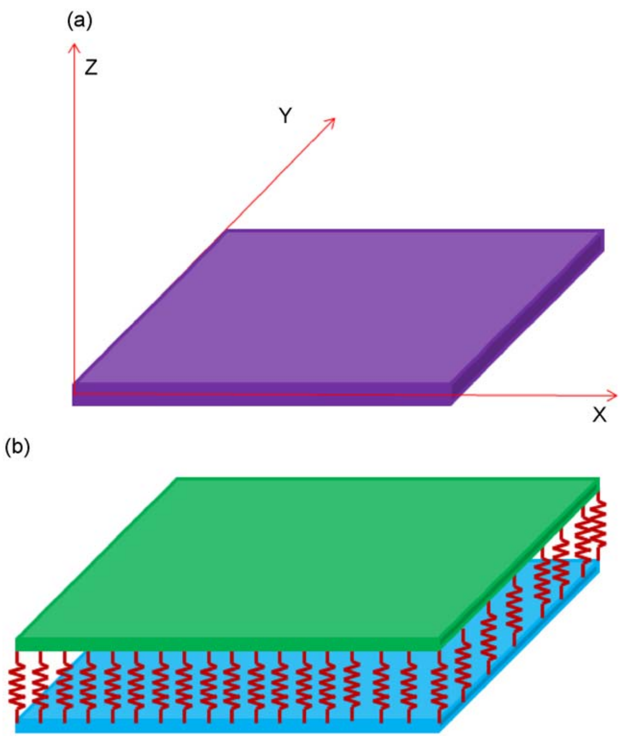

Fig. 1. Schematic of: (a) single layered and (b) double layered nanoplates.

Here μ is the nonlocal parameter and σ^l the local stress tensor at a point which is related to strain by generalized Hooke's law

$$\sigma^l(x) = S(x) : \varepsilon(x) \tag{3}$$

where S is the fourth-order elasticity tensor and ':' denotes the double dot product.

2.1. Classical plate theory (CLPT)

2.1.1. Single layered plate

A typical single layer plate is shown in Fig. 1a. CLPT for the single layered plate is based on the following displacement field

$$u_x = u(x,y,t) - z\frac{\partial w^c}{\partial x}, \quad u_y = v(x,y,t) - z\frac{\partial w^c}{\partial y} \quad \text{and} \quad u_z = w^c(x,y,t) \tag{4}$$

Here u, v and w^c denote displacement of the point (x, y, 0) along x, y and z directions, respectively.

The strains are expressed as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w^c}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial x} - z\frac{\partial^2 w^c}{\partial y^2}, \quad \varepsilon_{zz} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w^c}{\partial xy}\right),$$

$$\varepsilon_{xx} = 0, \quad \varepsilon_{yz} = 0 \tag{5}$$

It can be seen from Eq. (2) that nonlocal behavior enters into the problem through the constitutive relations. Principal of virtual work is independent of constitutive relations. So this can be applied to derive the equilibrium equations of the nonlocal plates. Using the principle of virtual displacements, following governing equations can be obtained [22]:

$$\frac{\partial N_0^{xx}}{\partial x} + \frac{\partial N_0^{xy}}{\partial y} = m_0\frac{\partial^2 u}{\partial t^2} \tag{6.1}$$

$$\frac{\partial N_0^{yy}}{\partial y} + \frac{\partial N_0^{xy}}{\partial x} = m_0\frac{\partial^2 v}{\partial t^2} \tag{6.2}$$

$$\frac{\partial^2 M_1^{xx}}{\partial y} + 2\frac{\partial^2 M_1^{xy}}{\partial x\partial y} + \frac{\partial^2 M_1^{yy}}{\partial y^2} + q + \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^c}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^c}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^c}{\partial y}\right)$$
$$+ \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^c}{\partial x}\right) = m_0\frac{\partial^2 w^c}{\partial t^2} - m_2\left(\frac{\partial^4 w^c}{\partial x^2\partial t^2} + \frac{\partial^4 w^c}{\partial y^2\partial t^2}\right) \tag{6.3}$$

m_0 and m_2 are mass moments of inertia and are defined as follows:

$$m_0 = \int_{-h/2}^{h/2} \rho\,\mathrm{d}z, \quad m_2 = \int_{-h/2}^{h/2} \rho h^2\,\mathrm{d}z \tag{7}$$

Here ρ denotes the density of the material. In CLPT, transverse shear stresses are neglected. Using Eq. (2), the plane stress constitutive relation for a nonlocal plate is written as

$$\begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} - \mu\nabla^2 \begin{Bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{Bmatrix} = \begin{bmatrix} E/(1-\upsilon^2) & \upsilon E/(1-\upsilon^2) & 0 \\ \upsilon E/(1-\upsilon^2) & E/(1-\upsilon^2) & 0 \\ 0 & 0 & 2G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{8}$$

E, G and v denote elastic modulus, shear modulus and Poisson's ratio, respectively.Using strain displacement relationship (Eq. (5)), stress–strain relationship (Eq. (8)) and stress resultants definition (Eq. (1)), we can express stress resultants in terms of displacements as follows:

$$M_1^{xx} - \mu\nabla^2 M_1^{xx} = -D\left(\frac{\partial^2 w^c}{\partial x^2} + \upsilon\frac{\partial^2 w^c}{\partial y^2}\right) \tag{9.1}$$

$$M_1^{yy} - \mu\nabla^2 M_1^{yy} = -D\left(\frac{\partial^2 w^c}{\partial y^2} + \upsilon\frac{\partial^2 w^c}{\partial x^2}\right) \tag{9.2}$$

$$M_1^{xy} - \mu\nabla^2 M_1^{xy} = -D(1-\upsilon)\frac{\partial^2 w^c}{\partial xy} \tag{9.3}$$

Here $D = Eh^3/12(1-\upsilon^2)$ denotes the bending rigidity of the plate. Using Eqs. (6.3) and (9) we get the following governing equations in terms of the displacements:

$$\begin{aligned}
&-D\nabla^4 w^c + \mu\nabla^2\left[-q - \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^c}{\partial x}\right) - \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^c}{\partial y}\right) - \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^c}{\partial y}\right) - \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^c}{\partial x}\right) + m_0\frac{\partial^2 w^c}{\partial t^2}\right.\\
&\left.- m_2\left(\frac{\partial^4 w^c}{\partial x^2\partial t^2} + \frac{\partial^4 w^c}{\partial y^2\partial t^2}\right)\right] + q + \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^c}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^c}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^c}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^c}{\partial x}\right)\\
&= m_0\frac{\partial^2 w^c}{\partial t^2} - m_2\left(\frac{\partial^4 w^c}{\partial x^2\partial t^2} + \frac{\partial^4 w^c}{\partial y^2\partial t^2}\right)
\end{aligned} \tag{10}$$

It can be noted that governing equation for traditional local CLPT can be obtained by setting $\mu = 0$ in Eq. (10).

2.1.2. Double layered plate

Nanobeams or nanoplates are generally found in the form of multilayered structures where two or more beam/plate layers are bonded by van der Walls (vdW) interaction. Modeling the vdW interaction is a key step in the continuum modeling of nanobeams/nanoplates. For multi walled nanobeam, this modeling has been discussed by various researchers [23–25]. Among these models most accepted one is the Winkler type foundation model for vdW forces. Kitipornchai et al. [20] extended the model for vdW forces to study vibration behavior of graphene sheets. They considered graphene sheets to be multilayered thin plates. In the present work this idea has been extended to nonlocal elastic double layered plates. A typical double layered plate is shown in Fig. 1b. Interaction between foundation parameter (the vdW force) and the nonlocal parameter is investigated.

Assuming that the distributed load is directly applied to the plate corresponding to displacement w_1^c (upper plate) the new distributed forces on these plates becomes

$$\begin{aligned}
q^1(x) &= q(x) - C(w_1^c - w_2^c)\\
q^2(x) &= -C(w_2^c - w_1^c)
\end{aligned} \tag{11}$$

Here superscript 1 and 2 in q correspond to plates with displacement w_1^c (upper plate) and w_2^c (lower plate), respectively. C can be calculated for a graphene sheet using the following formula [20]:

$$C_{graphene} = -\left(\frac{4\sqrt{3}}{9a}\right)^2 \frac{24\hbar}{\lambda^2}\left(\frac{\lambda}{a_{cc}}\right)^8 \frac{3003\pi}{256}\left\{\sum_{k=0}^{5}\frac{(-1)}{2k+1}\binom{5}{k}\left(\frac{\lambda}{a_{cc}}\right)^6\frac{1}{h_p^{12}} - \frac{35\pi}{8}\sum_{k=0}^{2}\frac{(-1)^k}{2k+1}\binom{2}{k}\frac{1}{h_p^6}\right\} \tag{12}$$

Here a_{cc} is the carbon–carbon bond length. h_p is the height of individual plates. $\hbar$ and λ are parameters that are chosen to fit the physical properties of the material.

Substituting q by $q^1(x)$ and $q^2(x)$ in Eq. (10) we get governing equations for double layered nanoplates

$$\begin{aligned}
&-D\nabla^4 w_1^c + \mu\nabla^2\left[-q + C(w_1^c - w_2^c) + \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_1^c}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_1^c}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_1^c}{\partial y}\right)\right.\\
&\left.+ \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_1^c}{\partial x}\right) + m_0\frac{\partial^2 w_1^c}{\partial t} - m_2\left(\frac{\partial^4 w_1^c}{\partial x^2\partial t^2} + \frac{\partial^4 w_1^c}{\partial y^2\partial t^2}\right)\right] + q - C(w_1^c - w_2^c) - \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_1^c}{\partial x}\right)\\
&- \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_1^c}{\partial y}\right) - \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_1^c}{\partial y}\right) - \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_1^c}{\partial x}\right) = m_0\frac{\partial^2 w_1^c}{\partial t^2} - m_2\left(\frac{\partial^4 w_1^c}{\partial x^2\partial t^2} + \frac{\partial^4 w_1^c}{\partial y^2\partial t^2}\right)
\end{aligned} \tag{13.1}$$

$$
\begin{aligned}
&- D\nabla^4 w_2^c + \mu\nabla^2\left[+C(w_2^c - w_1^c) + \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_2^c}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_2^c}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_2^c}{\partial y}\right)\right.\\
&\left.+\frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_2^c}{\partial x}\right) + m_0\frac{\partial^2 w_2^c}{\partial t} - m_2\left(\frac{\partial^4 w_2^c}{\partial x^2\partial t^2} + \frac{\partial^4 w_2^c}{\partial y^2\partial t^2}\right)\right] - C(w_2^c - w_1^c) - \frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_2^c}{\partial x}\right)\\
&-\frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_2^c}{\partial y}\right) - \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_2^c}{\partial y}\right) - \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_2^c}{\partial x}\right) = m_0\frac{\partial^2 w_2^c}{\partial t^2} - m_2\left(\frac{\partial^4 w_2^c}{\partial x^2\partial t^2} + \frac{\partial^4 w_2^c}{\partial y^2\partial t^2}\right)
\end{aligned} \tag{13.2}
$$

First and second equations correspond to displacements of upper (w_1^c) and lower (w_2^c) nanoplates, respectively.

2.2. *First-order shear deformation plate theory (FSDT)*

2.2.1. *Single layered plate*

According to FSDT the displacement field for single layered plate can be written as [22]

$$u_x = u(x,y,t) + z\psi_x(x,y,t), \quad u_y = v(x,y,t) + z\psi_y(x,y,t), \quad u_z = w^f(x,y,t) \tag{14}$$

Here u, v and w^f denote displacement at the point (x, y, 0) along x, y and z directions, respectively. ψ_x and ψ_y are the rotations of a transverse normal in the single layered plate with respect to x- and y-axis, respectively.

The strains are calculated as follows:

$$
\begin{aligned}
&\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial y}{\partial x} + z\frac{\partial \psi_y}{\partial y}, \quad \varepsilon_{zz} = 0, \quad \varepsilon_{yy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial x} + z\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right),\\
&\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \psi_x\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \psi_y\right)
\end{aligned} \tag{15}
$$

Using the principle of virtual displacements, following governing equations are obtained [22]

$$\frac{\partial N_0^{xx}}{\partial x} + \frac{\partial N_0^{xy}}{\partial y} = m_0\frac{\partial^2 u}{\partial t^2} \tag{16.1}$$

$$\frac{\partial N_0^{yy}}{\partial y} + \frac{\partial N_0^{xy}}{\partial x} = m_0\frac{\partial^2 v}{\partial t^2} \tag{16.2}$$

$$
\begin{aligned}
&\frac{\partial V_0^{xx}}{\partial x} + \frac{\partial V_0^{yy}}{\partial y} + q + \left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^f}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^f}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^f}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^f}{\partial x}\right)\right\}\\
&= m_0\frac{\partial^2 w_f}{\partial t^2}
\end{aligned} \tag{16.3}
$$

$$\frac{\partial M_1^{xx}}{\partial y} + \frac{\partial M_1^{xy}}{\partial x} - V_0^{xx} = m_2\frac{\partial^2 \psi_x}{\partial t^2} \tag{16.4}$$

$$\frac{\partial M_1^{yy}}{\partial y} + \frac{\partial M_1^{xy}}{\partial x} - V_0^{yy} = m_2\frac{\partial^2 \psi_y}{\partial t^2} \tag{16.5}$$

In FSDT, transverse shear stresses are taken into account. Using Eq. (2), the plane stress constitutive relation of a nonlocal plate with FSDT are expressed as in Eq. (8) and

$$\begin{Bmatrix} \sigma_{yz}^{nl} \\ \sigma_{xz}^{nl} \end{Bmatrix} - \mu\nabla^2\begin{Bmatrix} \sigma_{yz}^{nl} \\ \sigma_{xz}^{nl} \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} \tag{17}$$

Using strain displacement relationship (Eq. (15)), stress–strain relationship (Eqs. (8) and (17)) and stress resultants definitions (Eq. (1)), one can express stress resultants in terms of displacements as follows:

$$M_1^{xx} - \mu\nabla^2 M_1^{xx} = D\left(\frac{\partial\psi_x}{\partial x} + \upsilon\frac{\partial\psi_y}{\partial y}\right) \tag{18.1}$$

$$M_1^{yy} - \mu\nabla^2 M_1^{yy} = D\left(\frac{\partial\psi_y}{\partial y} + \upsilon\frac{\partial\psi_x}{\partial x}\right) \tag{18.2}$$

$$M_1^{xy} - \mu\nabla^2 M_1^{xy} = \frac{1}{2}D(1-\upsilon)\left(\frac{\partial\psi_x}{\partial y} + \upsilon\frac{\partial\psi_y}{\partial x}\right) \tag{18.3}$$

$$V_0^{xx} - \mu\nabla^2 V_0^{xx} = \kappa^2 Gh\left(\psi_x + \frac{\partial w}{\partial x}\right) \tag{18.4}$$

$$V_0^{yy} - \mu\nabla^2 V_0^{yy} = \kappa^2 Gh\left(\psi_y + \frac{\partial w}{\partial y}\right) \tag{18.5}$$

In the above equations, κ^2 denote the shear correction factor. Using Eqs. (16) and (18) we get the following governing equations in terms of displacements:

$$\begin{aligned}
&\kappa^2 Gh\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w^f}{\partial x^2} + \frac{\partial^2 w^f}{\partial y^2}\right) + q + \left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^f}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^f}{\partial y}\right)\right.\\
&\quad\left.+\frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^f}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^f}{\partial x}\right)\right\} - \mu\nabla^2\left[q + \left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w^f}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w^f}{\partial y}\right)\right.\right.\\
&\quad\left.\left.+\frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w^f}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w^f}{\partial x}\right)\right\}\right] = m_0\left(\frac{\partial^2 w}{\partial t^2} - \mu\nabla^2\frac{\partial^2 w}{\partial t^2}\right)
\end{aligned} \tag{19.1}$$

$$D\left[\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{2}(1-\upsilon)\frac{\partial^2\psi_x}{\partial y^2} + \frac{1}{2}(1+\upsilon)\frac{\partial^2\psi_y}{\partial x\partial y}\right] - \kappa^2 Gh\left(\psi_x + \frac{\partial w^f}{\partial x}\right) = m_2\left(\frac{\partial^2\psi_x}{\partial t^2} - \mu\nabla^2\frac{\partial^2\psi_x}{\partial t^2}\right) \tag{19.2}$$

$$D\left[\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{2}(1-\upsilon)\frac{\partial^2\psi_y}{\partial x^2} + \frac{1}{2}(1+\upsilon)\frac{\partial^2\psi_y}{\partial x\partial y}\right] - \kappa^2 Gh\left(\psi_y + \frac{\partial w^f}{\partial y}\right) = m_2\left(\frac{\partial^2\psi_y}{\partial t^2} - \mu\nabla^2\frac{\partial^2\psi_y}{\partial t^2}\right) \tag{19.3}$$

2.2.2. *Double layered plate*

Making use of the same argument as in CLPT (Section 2.1.2) and substituting q by $q^1(x)$ and $q^2(x)$ in Eq. (19) we get governing equations for double layered nanoplates

$$\begin{aligned}
&\kappa^2 Gh\left(\frac{\partial\psi 1_x}{\partial x} + \frac{\partial\psi 1_y}{\partial y} + \frac{\partial^2 w_1^f}{\partial x^2} + \frac{\partial^2 w_1^f}{\partial y^2}\right) - C(w_1^f - w_2^f) + \left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_1^f}{\partial x}\right)\right.\\
&\quad\left.+\frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_1^f}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{yy}\frac{\partial w_1^f}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_1^f}{\partial x}\right)\right\} - \mu\nabla^2\left[-C(w_1^f - w_2^f)\right.\\
&\quad\left.-\left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_1^f}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_1^f}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_1^f}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_1^f}{\partial x}\right)\right\}\right]\\
&\quad= m_0\left(\frac{\partial^2 w_1^f}{\partial t} - \mu\nabla^2\frac{\partial^2 w_1^f}{\partial t^2}\right)
\end{aligned} \tag{20.1}$$

$$D\left[\frac{\partial^2\psi 1_x}{\partial x^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2\psi 1_x}{\partial y^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2\psi 1_y}{\partial x\partial y}\right]-\kappa^2 Gh\left(\psi 1_x+\frac{\partial w_1^f}{\partial x}\right)$$
$$=m_2\left(\frac{\partial^2\psi 1_x}{\partial t^2}-\mu\nabla^2\frac{\partial^2\psi 1_x}{\partial t^2}\right) \tag{20.2}$$

$$D\left[\frac{\partial^2\psi 1_y}{\partial y^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2\psi 1_y}{\partial x^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2\psi 1_x}{\partial x\partial y}\right]-\kappa^2 Gh\left(\psi 1_y+\frac{\partial w_1^f}{\partial y}\right)$$
$$=m_2\left(\frac{\partial^2\psi 1_y}{\partial t^2}-\mu\nabla^2\frac{\partial^2\psi 1_y}{\partial t^2}\right) \tag{20.3}$$

$$\kappa^2 Gh\left(\frac{\partial\psi 2_x}{\partial x}+\frac{\partial\psi 2_y}{\partial y}+\frac{\partial^2 w_2^f}{\partial x^2}+\frac{\partial^2 w_2^f}{\partial y^2}\right)-C(w_2^f-w_1^f)+\left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_2^f}{\partial x}\right)\right.$$
$$\left.+\frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_2^f}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_0^{yy}\frac{\partial w_2^f}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_2^f}{\partial x}\right)\right\}-\mu\nabla^2\left[-C(w_2^f-w_1^f)\right.$$
$$\left.-\left\{\frac{\partial}{\partial x}\left(N_0^{xx}\frac{\partial w_2^f}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_0^{yy}\frac{\partial w_2^f}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_0^{xy}\frac{\partial w_2^f}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_0^{xy}\frac{\partial w_2^f}{\partial x}\right)\right\}\right]$$
$$=m_0\left(\frac{\partial^2 w_2^f}{\partial t}-\mu\nabla^2\frac{\partial^2 w_2^f}{\partial t^2}\right) \tag{20.4}$$

$$D\left[\frac{\partial^2\psi 2_x}{\partial x^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2\psi 2_x}{\partial y^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2\psi 2_y}{\partial x\partial y}\right]-\kappa^2 Gh\left(\psi 2_x+\frac{\partial w_2^f}{\partial x}\right)$$
$$=m_2\left(\frac{\partial^2\psi 2_x}{\partial t^2}-\mu\nabla^2\frac{\partial^2\psi 2_x}{\partial t^2}\right) \tag{20.5}$$

$$D\left[\frac{\partial^2\psi 2_y}{\partial y^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2\psi 2_y}{\partial x^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2\psi 2_x}{\partial x\partial y}\right]-\kappa^2 Gh\left(\psi 2_y+\frac{\partial w_2^f}{\partial y}\right)$$
$$=m_2\left(\frac{\partial^2\psi 2_y}{\partial t^2}-\mu\nabla^2\frac{\partial^2\psi 2_y}{\partial t^2}\right) \tag{20.6}$$

In the above equations, w_1^f, $\psi 1_x$ and $\psi 1_y$ denote deflection, rotation of the normal with respect to x-axis and rotation of the normal with respect to y-axis, respectively for the upper plate. Similarly, w_2^f, $\psi 2_x$ and $\psi 2_y$ denote deflection, rotation of the normal with respect to *x-axis* and rotation of the normal with respect to y-axis, respectively for the lower plate.

3. Solution using Navier's approach

The developed governing differential equations of Section 2 have been solved by Navier's approach for simply supported boundary conditions. The simply supported boundary conditions for CLPT and FSDT are written as

CLPT
At $x=0$ and $x=a$
$u=0,\ v=0$
$N_0^{xx}=0,\ M_1^{xx}=0$

FSDT
At $x=0$ and $x=a$
$u=0,\ v=0,\ \psi_y=0$
$N_0^{xx}=0,\ M_1^{xx}=0$

At $y=0$ and $y=b$
$u=0,\ v=0$
$N_0^{yy}=0,\ M_1^{yy}=0$

At $y=0$ and $y=b$
$u=0,\ v=0,\ \psi_x=0$
$N_0^{yy}=0,\ M_1^{yy}=0$

The following expressions of various generalized displacements have been assumed:

$$w^c=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}^c\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega_{mn}^c t} \tag{21.1}$$

$$w^f=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}^f\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega_{mn}^f t} \tag{21.2}$$

$$w_1^c=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{1mn}^c\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^c t} \tag{21.3}$$

$$w_2^c=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{2mn}^c\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^c t} \tag{21.4}$$

$$w_1^f=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{1mn}^f\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.5}$$

$$w_2^f=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{2mn}^f\ \sin(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.6}$$

$$\psi_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_{mn}\ \cos(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega_{mn}^f t} \tag{21.7}$$

$$\psi_y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{mn}\ \sin(\alpha x)\cos(\beta y)\mathrm{e}^{\mathrm{i}\omega_{mn}^f t} \tag{21.8}$$

$$\psi 1_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X1_{mn}\ \cos(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.9}$$

$$\psi 2_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X2_{mn}\ \cos(\alpha x)\sin(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.10}$$

$$\psi 1_y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y1_{mn}\ \sin(\alpha x)\cos(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.11}$$

$$\psi 2_y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y2_{mn}\ \sin(\alpha x)\cos(\beta y)\mathrm{e}^{\mathrm{i}\omega^f t} \tag{21.12}$$

In the above expressions, $\alpha=m\pi/a$ and $\beta=n\pi/b$.

3.1. Classical plate theory (CLPT)

It is assumed that the plate is free from any in-plane or transverse loadings. So we have

$$N_0^{xx}=N_0^{yy}=N_0^{xy}=q=0$$

Substituting Eq. (21.1) into Eq. (10) we get,

$$-D(\alpha^2+\beta^2)^2 W^c_{mn} = -M_{mn}\lambda_{mn}{\omega^c_{mn}}^2 W^c_{mn} \tag{22}$$

Here $\lambda_{mn} = 1+\mu(\alpha^2+\beta^2)$. Using Eq. (22) following natural frequencies are obtained

$$\omega^c_{mn} = \sqrt{\frac{D(\alpha^2+\beta^2)^2}{M_{mn}\lambda_{mn}}} \tag{23}$$

Here $M_{mn} = m_0 + m_2(\alpha^2+\beta^2)$. It can be seen from Eq. (23) that increase in nonlocal parameter would decrease the natural vibration frequencies.

Substituting Eq. (21.3) and Eq. (21.4) into Eq. (13) we get

$$\begin{aligned} &-D(\alpha^2+\beta^2)^2 W^c_{1mn} - C\lambda_{mn}(W^c_{1mn} - W^c_{2mn}) = -M_{mn}\lambda_{mn}\omega^{c2} W^c_{1mn} \\ &-D(\alpha^2+\beta^2)^2 W^c_{2mn} - C\lambda_{mn}(W^c_{2mn} - W^c_{1mn}) = -M_{mn}\lambda_{mn}\omega^{c2} W^c_{2mn} \end{aligned} \tag{24}$$

Natural frequencies are obtained from following expression:

$$\det\begin{bmatrix} -D(\alpha^2+\beta^2)^2 - C\lambda_{mn} + M_{mn}\lambda_{mn}\omega^2 & C\lambda_{mn} \\ C\lambda_{mn} & -D(\alpha^2+\beta^2)^2 - C\lambda_{mn} + M_{mn}\lambda_{mn}\omega^2 \end{bmatrix} = 0 \tag{25}$$

Solving for ω we get

$$\begin{aligned} \omega^c_{mn}1 &= \sqrt{\frac{D(\alpha^2+\beta^2)^2}{M_{mn}\lambda_{mn}}} \\ \omega^c_{mn}2 &= \sqrt{\frac{D(\alpha^2+\beta^2)^2 + 2C\lambda_{mn}}{M_{mn}\lambda_{mn}}} \end{aligned} \tag{26}$$

It can be seen that one set of the frequencies are independent of vdW forces but other set of frequencies are dependent on the vdW forces.

3.2. First-order shear deformation theory (FSDT)

Substituting Eqs. (21.2), (21.7) and (21.8) into Eq. (19) we get

$$\begin{aligned} &\{-(\alpha^2+\beta^2)\kappa^2 Gh + \omega^{f2}_{mn}\lambda_{mn}\} W^f_{mn} - \alpha\kappa^2 Gh X_{mn} - \beta\kappa^2 Gh Y_{mn} = 0 \\ &-\alpha\kappa^2 Gh W^f_{mn} - [D\{\alpha^2 + \tfrac{1}{2}(1-\upsilon)\beta^2\} + \kappa^2 Gh - \omega^{f2}_{mn} m_2\lambda_{mn}] X_{mn} - \frac{D\alpha\beta}{2}(1+\upsilon) Y_{mn} = 0 \\ &-\beta\kappa^2 Gh W^f_{mn} - \frac{D\alpha\beta}{2}(1+\upsilon) X_{mn} - [D\{\beta^2 + \tfrac{1}{2}(1-\upsilon)\alpha^2\} + \kappa^2 Gh - \omega^{f2}_{mn} m_2\lambda_{mn}] Y_{mn} = 0 \end{aligned} \tag{27}$$

One can rewrite Eq. (27) as

$$[\bar{\boldsymbol{S}}_s] + \omega^{f2}_{\boldsymbol{mn}}\lambda_{\boldsymbol{mn}}[\bar{\boldsymbol{G}}_s][\bar{\boldsymbol{D}}_s] = [0\ 0\ 0]^{\mathrm{T}} \tag{28}$$

$[\bar{\boldsymbol{S}}_s]$, $[\bar{\boldsymbol{G}}_s]$ and $[\bar{\boldsymbol{D}}_s]$ are defined in Appendix A.

Natural frequencies are obtained from Eq. (28)

Neglecting rotary inertia and solving Eq. (28) we get the following closed form solution for natural vibration frequency of nonlocal plates.

$$\omega^f_{mn} = \sqrt{\frac{\gamma_1(\gamma_4\gamma_6 - \gamma_5^2) - \gamma_2(\gamma_2\gamma_6 - \gamma_3\gamma_5) + \gamma_3(\gamma_2\gamma_5 - \gamma_3\gamma_4)}{m_0\lambda_{mn}(\gamma_5^2 - \gamma_4\gamma_6)}} \tag{29}$$

Substituting Eqs. (21.5), (21.6) and (21.9)–(21.12) into Eq. (19) we get

$$\{-(\alpha^2+\beta^2)\kappa^2 Gh+\omega^{f2}\lambda_{mn}-C\lambda_{mn}\}W^f_{1mn}-\alpha\kappa^2 GhX1_{mn}-\beta\kappa^2 GhY1_{mn}+C\lambda_{mn}W^f_{2mn}=0$$
$$-\alpha\kappa^2 GhW^f_{1mn}-[D\{\alpha^2+\tfrac{1}{2}(1-\upsilon)\beta^2\}+\kappa^2 Gh-\omega^{f2}m_2\lambda_{mn}]X1_{mn}-\frac{D\alpha\beta}{2}(1+\upsilon)Y1_{mn}=0$$
$$-\beta\kappa^2 GhW^f_{1mn}-\frac{D\alpha\beta}{2}(1+\upsilon)X1_{mn}-[D\{\beta^2+\tfrac{1}{2}(1-\upsilon)\alpha^2\}+\kappa^2 Gh-\omega^{f2}m_2\lambda_{mn}]Y1_{mn}=0$$
$$\{-(\alpha^2+\beta^2)\kappa^2 Gh+\omega^{f2}\lambda_{mn}-C\lambda_{mn}\}W^f_{2mn}-\alpha\kappa^2 GhX2_{mn}-\beta\kappa^2 GhY2_{mn}+C\lambda_{mn}W^f_{1mn}=0$$
$$-\alpha\kappa^2 GhW^f_{2mn}-[D\{\alpha^2+\tfrac{1}{2}(1-\upsilon)\beta^2\}+\kappa^2 Gh-\omega^{f2}m_2\lambda_{mn}]X2_{mn}-\frac{D\alpha\beta}{2}(1+\upsilon)Y2_{mn}=0$$
$$-\beta\kappa^2 GhW^f_{2mn}-\frac{D\alpha\beta}{2}(1+\upsilon)X1_{mn}-[D\{\beta^2+\tfrac{1}{2}(1-\upsilon)\alpha^2\}+\kappa^2 Gh-\omega^{f2}m_2\lambda_{mn}]Y1_{mn}=0 \tag{30}$$

Eq. (30) is rewritten as

$$[\bar{\boldsymbol{S}}_{\boldsymbol{d}}]+\omega^{f2}_{\boldsymbol{mn}}\lambda_{\boldsymbol{mn}}[\bar{\boldsymbol{G}}_{\boldsymbol{d}}][\bar{\boldsymbol{D}}_{\boldsymbol{d}}]=[0\;0\;0\;0\;0\;0]^{\mathrm{T}} \tag{31}$$

$[\bar{\boldsymbol{S}}_{\boldsymbol{d}}]$, $[\bar{\boldsymbol{G}}_{\boldsymbol{d}}]$ and $[\bar{\boldsymbol{D}}_{\boldsymbol{d}}]$ are defined in Appendix A. Neglecting rotary inertia and using Eq. (31) following expressions for natural frequencies are obtained.

$$\omega^{f}_{mn}1=\sqrt{\frac{\gamma_1'(\gamma_4\gamma_6-\gamma_5^2)-\gamma_2(\gamma_2\gamma_6-\gamma_3\gamma_5)+\gamma_3(\gamma_2\gamma_5-\gamma_3\gamma_4)+\gamma_4\gamma_6\gamma_7-\gamma_7\gamma_5^2}{m_0\lambda_{mn}(\gamma_5^2-\gamma_4\gamma_6)}}$$

$$\omega^{f2}_{mn}=\sqrt{\frac{\gamma_1'(\gamma_4\gamma_6-\gamma_5^2)-\gamma_2(\gamma_2\gamma_6-\gamma_3\gamma_5)+\gamma_3(\gamma_2\gamma_5-\gamma_3\gamma_4)-\gamma_4\gamma_6\gamma_7+\gamma_7\gamma_5^2}{m_0\lambda_{mn}(\gamma_5^2-\gamma_4\gamma_6)}} \tag{32}$$

As observed in CLPT case (Section 3.1) here in FSDT case one finds that one set of frequencies is independent of vdW forces. While other set is dependent on vdW forces.

4. Results and discussions

The governing equations for vibration of nonlocal plates are written in Eqs. (10) and (19). It can be seen that putting $\mu=0$ in these equations traditional local elastic plate vibration equations are obtained. These governing equations for local elasticity theory are same as expressed in Reddy [22]. Further using nonlocal elasticity theory one could derive the governing equation Eq. (13) for vibration of multilayered plates. It can be noted that by setting $\mu=0$, this equation can obtain the corresponding local elasticity equation. This derived local elasticity equation matches with that of Kitipornchai's et al. [20]. Further, putting $D=EI$ and $b=\infty$ in Eqs. (24) and (30) nonlocal solutions for free vibration of beam are obtained. These derived equations do match with the nonlocal solutions for free vibration of beam (Reddy [16]). A beam with following material properties and geometrical dimensions are considered: elastic modulus $E_b=30\,\text{GPa}$, length $L_b=10\,\text{m}$, height $h_b=$ varied, density $\rho_b=1\,\text{kg/m}^3$. Natural frequencies are nondimensionalized as

$$\bar{\omega}_b=\omega_b\times L_b^2\sqrt{\frac{\rho_b h_b}{E_b I_b}}$$

Nondimensional natural frequencies using Eqs. (24) and (30) are calculated for the above mentioned beam. These results are listed in Table 1. From this table one could find that present results for the beam exactly match with those reported by Reddy [16].

It can be seen from Eqs. (24) and (30) that for single layered nanoplates, the percentage difference in free vibration frequencies calculated using local and nonlocal elasticity theory will depend on (i) size (length or breadth) of the plate, (ii) mode of vibration and (iii) nonlocal parameter. This is true for both CLPT and FSDT. In the present work the plate is considered to be a square plate. Frequency ratio is defined as the ratio of the frequency obtained using nonlocal elasticity theory to frequency obtained using local elasticity theory ($\mu=0$).

Table 1
Nondimensional natural frequencies for EBT and TBT of beams.

L/h	μ	Nondimensional natural frequency from EBT [16]	Nondimensional natural frequency from EBT (present)	Nondimensional natural frequency from TBT [16]	Nondimensional natural frequency from TBT (present)
100	0.0	9.8696	9.8696	9.8683	9.8683
	0.5	9.6347	9.6347	9.6335	9.6335
	1.0	9.4159	9.4159	9.4147	9.4147
	1.5	9.2113	9.2113	9.2101	9.2101
	2.0	9.0195	9.0195	9.0183	9.0183
20	0.0	9.8696	9.8696	9.8381	9.8381
	0.5	9.6347	9.6347	9.6040	9.6040
	1.0	9.4159	9.4159	9.3858	9.3858
	1.5	9.2113	9.2113	9.1819	9.1819
	2.0	9.0195	9.0195	8.9907	8.9907

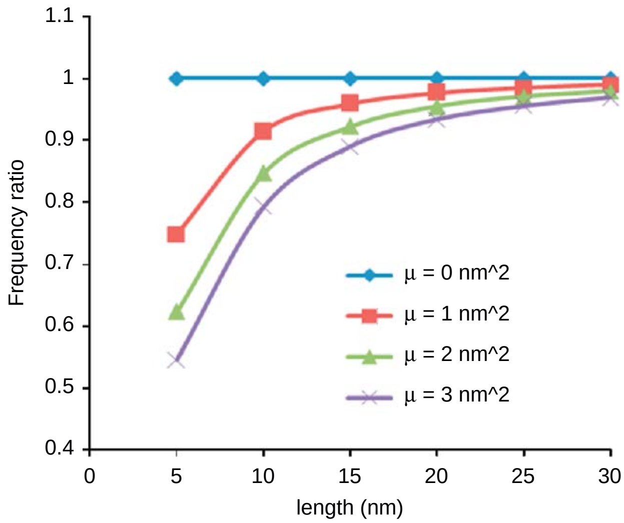

Fig. 2. Variation of natural frequencies ratio c with the length of a square nanoplate for various nonlocal parameter.

For various nonlocal parameters and lengths of the plates the frequency ratios are plotted in Fig. 2. This figure shows the profound scale effect for smaller size plate and higher values of nonlocal parameter. From this figure it can also be observed that lower frequency ratio is obtained at higher values of nonlocal parameter. Further it can be observed that as length increases, frequency ratio increases. This observation is attributed to the fact that nonlocal effect is more profound in the case of small nano lengths. Frequency ratio for various lengths of the plate and various modes of vibration are plotted in Fig. 3. The value of nonlocal parameter (μ) is assumed to be $2\,\text{nm}^2$. It can be seen that the frequency ratios decrease with increase in vibration modes. This reveals that nonlocal parameter is more prominent in higher vibration modes. From Eqs. (27) and (33) it can be seen that, two sets of natural frequencies are obtained for double layered plates. One set of frequencies are independent of vdW forces and are exactly same as those of single layered plate. Further it can be observed that for double layered plate, frequency ratio associated with second set depends on (i) size (length or breadth) of the plate, (ii) mode of vibration, (iii) nonlocal parameter, (iv) elastic modulus (E), (v) Poisson's ratio (v), (vi) height (h) and (vii) shear correction factor (κ^2). This dependency of frequency ratio on these parameters is attributed to the fact that to illustrate this nonlocal effect on double layered plate, a square shaped graphene sheet with following properties [19] is considered. FSDT has been employed in this

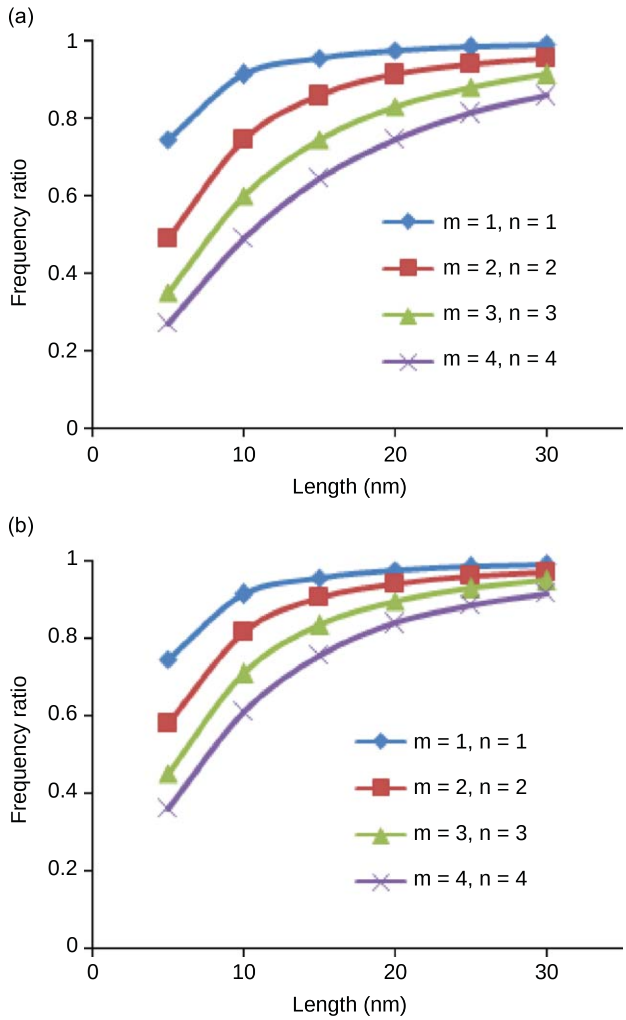

Fig. 3. Variation of natural frequencies ratio with length of a square plate for various modes of vibration: (a) $m = n$ and (b) $m! = n$.

numerical example. The elastic modulus $E = 1.02\,\text{TPa}$, length or breadth $L = 10\,\text{nm}$, thickness of each plate $h = 0.34\,\text{nm}$, the Poisson's ratio $v = 0.3$, Winkler foundation modulus $C = 108\,\text{GPa/nm}$ are assumed. First few modes of vibrations are considered and vibration frequency ratios have been plotted against percentage change of each variable, one at a time. These numerical results are shown in Figs. 4 and 5. Present nonlocal elasticity solutions show frequency ratio decreases with increase in height and Young's modulus and increases with length (or breadth) and Winkler modulus. This is more prominent for higher modes of vibrations (Fig. 5). To explain these trends we rewrite local and nonlocal CLPT results for double layered plates (Eq. (2), second equation) as follows:

$$\omega^2_{\text{local}} = \frac{D(\alpha^2 + \beta^2)^2 + 2C}{M_{mn}}, \quad \omega^2_{\text{non-local}} = \frac{\frac{1}{\lambda_{mn}} D(\alpha^2 + \beta^2)^2 + 2C}{M_{mn}}$$

As $\lambda_{mn} > 1$ for $\mu \neq 0$, increase in $D(\alpha^2 + \beta^2)^2$ will cause increase in $\omega^2_{\text{non-local}}$ at a slower rate than for ω^2_{local} and vice-versa. Thus from definition of D, α and β, increase in E and h will make the ratio $(\omega_{\text{non-local}}/\omega_{\text{local}})$ decrease and increase in L will make this ratio increase. To explain the effect of C, consider the numerator of $\omega^2_{\text{non-local}}$ which is less than the numerator of ω^2_{local}. This implies increase in C will make the ratio

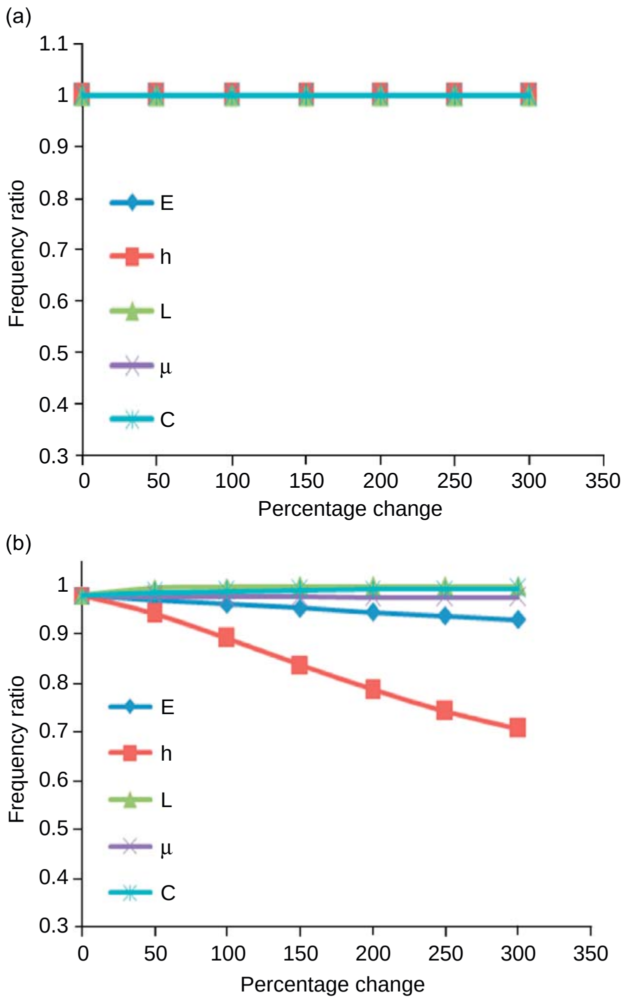

Fig. 4. Frequency ratio for double layered plate versus percentage variation of individual parameters for vibration modes: (a) $m = n = 1$ and (b) $m = n = 3$.

$(\omega_{\text{non-local}}/\omega_{\text{local}})$ increase and vice-versa. Similar procedure can be adopted to explain the trends using FSDT results.

The percentage difference of frequencies in using CLPT and FSDT in single layered plate has been defined as follows:

$$\text{Percentage difference} = \left|\frac{\omega_{mn}^{c} - \omega_{mn}^{f}}{\omega_{mn}^{f}}\right| \times 100$$

While the percentage difference of frequencies in using CLPT and FSDT in double layered plate has been defined as follows:

$$\text{Percentage difference} = \left|\frac{\omega_{mn}^{c}2 - \omega_{mn}^{f}2}{\omega_{mn}^{f}2}\right| \times 100$$

The percentage differences of the frequencies (employing CLPT and FSDT) for single layered and double layered plates are plotted in Fig. 6(a) and (b), respectively. It is interesting to note that the difference is significantly smaller for double layered plate than that for single layered plate. This is attributed to the fact

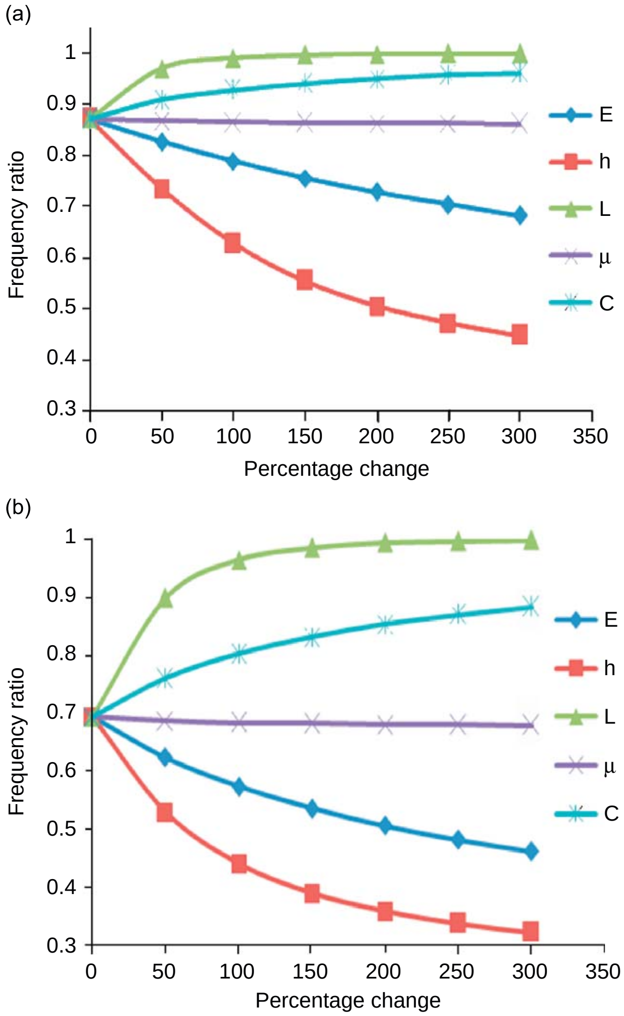

Fig. 5. Frequency ratio for double layered plate versus percentage variation of individual parameters for vibration modes: (a) $m = n = 5$ and (b) $m = n = 7$.

that in the formulation of double layered plate three layers of materials are employed viz. top layer, Winkler foundation and bottom layer. This takes into account the shear forces in the double layered plate. Thus in double layered plates there is less difference between CLPT and FSDT results.

5. Conclusions

Equations of motion of Classical plate theory and first-order shear deformation theory of the plates are derived based on Eringen's differential constitutive equations of nonlocal elasticity. The equations of motion are then analytically solved to obtain closed form solution for natural vibration of simply supported single layered and double layered plates. Effect of (i) nonlocal parameter, (ii) length, (iii) height, (iv) elastic modulus and (v) stiffness of Winkler foundation, of the plate on nondimensional vibration frequencies based on the nonlocal elasticity theory are investigated. Nondimensional natural frequencies decrease with increase in mode number. As the size of the plate decreases the effect of nonlocal theory becomes more significant and predicts smaller nondimensional natural frequencies.

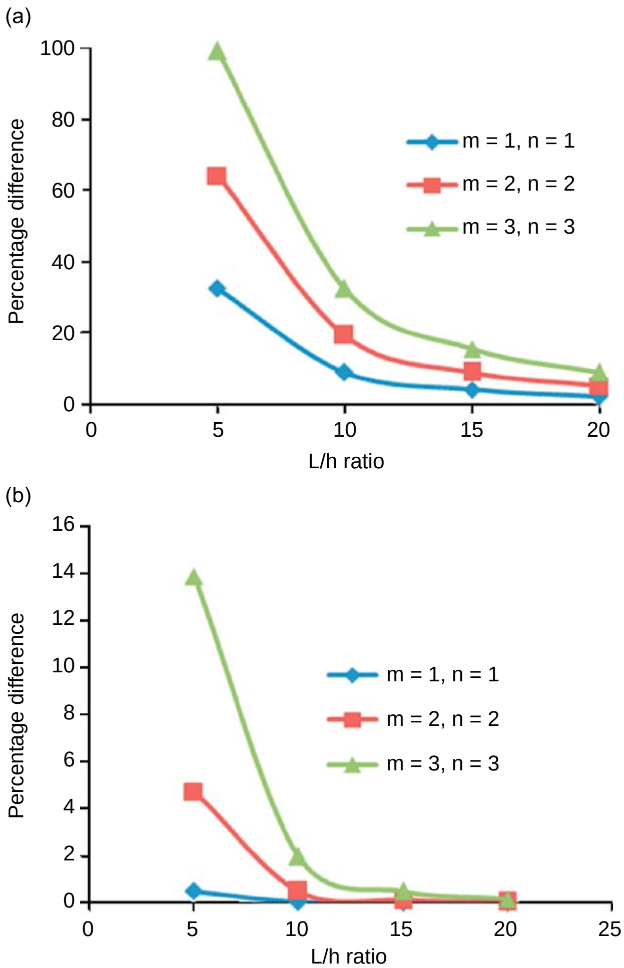

Fig. 6. Percentage difference in vibration frequency using CLPT and FSDT versus L/h ratio for: (a) single layered and (b) double layered nanoplates.

Present nonlocal elasticity solutions show nondimensional natural frequencies ratio decrease with increase in height and Young's modulus of the nanoplate and increase with length (or breadth of the plate) and Winkler modulus of the medium. This is more prominent for higher modes of vibrations. While insignificant changes are observed for nonlocal parameter.

Effect of first-order shear deformation theory with the nonlocal elasticity on nondimensional natural frequencies is found to be significant for thicker plates. Classical plate theory over predicts the natural frequency. CLPT and FSDT are employed and the frequency predicted for single layered and double layered plates. The difference in the frequencies predicted by CLPT and FSDT is significantly smaller for double layered plate than that for single layered plate.

Acknowledgment

Authors are grateful to Professor J.N. Reddy of Texas A&M University, US for the technical discussions about the Eringen's nonlocal elasticity theory.

Appendix A

$$[\bar{S}_s] = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_4 & \gamma_5 \\ \gamma_3 & \gamma_5 & \gamma_6 \end{bmatrix}$$

$$[\bar{G}_s] = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix}$$

$$[\bar{D}_s] = \begin{bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{bmatrix}$$

$$[\bar{S}_d] = \begin{bmatrix} \gamma_8 & \gamma_2 & \gamma_3 & \gamma_7 & 0 & 0 \\ \gamma_2 & \gamma_4 & \gamma_5 & 0 & 0 & 0 \\ \gamma_3 & \gamma_5 & \gamma_6 & 0 & 0 & 0 \\ \gamma_7 & 0 & 0 & \gamma_8 & \gamma_2 & \gamma_3 \\ 0 & 0 & 0 & \gamma_2 & \gamma_4 & \gamma_5 \\ 0 & 0 & 0 & \gamma_3 & \gamma_5 & \gamma_6 \end{bmatrix}$$

$$[\bar{G}_d] = \begin{bmatrix} m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix}$$

$$[\bar{D}_d] = \begin{bmatrix} W_1 \\ X1 \\ Y1 \\ W_2 \\ X2 \\ Y2 \end{bmatrix}$$

$$\gamma_1 = -(\alpha^2 + \beta^2)\kappa^2 Gh$$
$$\gamma_2 = -\alpha\kappa^2 Gh$$
$$\gamma_3 = -\beta\kappa^2 Gh$$
$$\gamma_4 = -D\{\alpha^2 + \tfrac{1}{2}(1 - \upsilon)\beta^2\} - \kappa^2 Gh$$

$$\gamma_5 = -\frac{D\alpha\beta}{2}(1+\upsilon)$$
$$\gamma_6 = -D\{\beta^2 + \tfrac{1}{2}(1-\upsilon)\alpha^2\} - \kappa^2 Gh$$
$$\gamma_7 = C\lambda_{mn}$$
$$\gamma_8 = -(\alpha^2+\beta^2)\kappa^2 Gh - C\lambda_{mn}$$

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